SIMPLE ZEROS OF AUTOMORPHIC L-FUNCTIONS

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Abstract. We prove that the complete L-function associated to any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ has infinitely many simple zeros.

1. Introduction

In [1], the first author showed that the complete L-functions associated to classical holomorphic newforms have infinitely many simple zeros. The purpose of this paper is to extend that result to the remaining degree 2 automorphic L-functions over $\mathbb{Q}$, i.e. those associated to cuspidal Maass newforms. This also extends work of the second author [4] which established a quantitative estimate for the first few Maass forms of level 1. When combined with the holomorphic case from [1], we obtain the following:

Theorem 1.1. *Let $\mathbb{A}_{\mathbb{Q}}$ denote the adèle ring of $\mathbb{Q}$, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Then the associated complete L-function $\Lambda(s,\pi)$ has infinitely many simple zeros.*

The basic idea of the proof is the same as in [1], which is in turn based on the method of Conrey and Ghosh [5]. Let f be a primitive Maass cuspform of weight $k \in \{0,1\}$ for $\Gamma_0(N)$ with nebentypus character ξ, and let $L_f(s)$ be the finite L-function attached to f:

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

We define

$$D_f(s) = L_f(s) \frac{d^2}{ds^2} \log L_f(s) = \sum_{n=1}^{\infty} c_f(n) n^{-s}.$$

Then it is easy to see that $D_f(s)$ has a pole at some point if and only if $L_f(s)$ has a simple zero there.

For $\alpha \in \mathbb{Q}$ and $j \geq 0$ we define the additive twists

$$L_f(s,\alpha,\cos^{(j)}) = \sum_{n=1}^{\infty} \lambda_f(n) \cos^{(j)}(2\pi n\alpha) n^{-s}, \quad D_f(s,\alpha,\cos^{(j)}) = \sum_{n=1}^{\infty} c_f(n) \cos^{(j)}(2\pi n\alpha) n^{-s},$$

where $\cos^{(j)}$ denotes the jth derivative of the cosine function. Let $q \nmid N$ be a prime and χ_0 the principal character mod q. Then we have the following expansions of the trigonometric functions in terms of Dirichlet characters:

$$\cos\left(\frac{2\pi n}{q}\right) = 1 - \frac{q}{q-1}\chi_0(n) + \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\epsilon_\chi} \chi(n),$$

$$\sin\left(\frac{2\pi n}{q}\right) = \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=-1}} \overline{\epsilon_\chi} \chi(n),$$

where ϵ_χ denotes the root number of the Dirichlet L-function $L(s,\chi)$. In particular, we have

$$D_f(s, \tfrac{1}{q}, \cos) = D_f(s) - \frac{q}{q-1} D_f(s,\chi_0) + \frac{\sqrt{q}}{q-1} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \overline{\epsilon_\chi} D_f(s,\chi),$$

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where

$$D_f(s,\chi) = \sum_{n=1}^{\infty} c_f(n)\chi(n)n^{-s}$$

is the corresponding multiplicative twist.

By the non-vanishing results for automorphic L-functions [8], all non-trivial poles of $D_f(s)$ and $D_f(s,\chi)$ for $\chi \neq \chi_0$ are located in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. However, for the case of the principal character, since

$$L_f(s,\chi_0) = \sum_{n=1}^{\infty} \lambda_f(n)\chi_0(n)n^{-s} = (1 - \lambda_f(q)q^{-s} + \xi(q)q^{-2s})L_f(s),$$

$D_f(s,\chi_0)$ has a pole at every simple zero of the local Euler factor polynomial, $1 - \lambda_f(q)q^{-s} + \xi(q)q^{-2s}$, at which $L_f(s)$ does not vanish.

Since f is cuspidal, the Rankin–Selberg method implies that the average of $|\lambda_f(q)|^2$ over primes q is 1, i.e.

$$\lim_{x\to\infty} \frac{\sum_{\substack{q \text{ prime} \\ q \le x}} |\lambda_f(q)|^2}{\#\{q \text{ prime} : q \le x\}} = 1. \tag{1.1}$$

To see this, write

$$-\frac{L_f'}{L_f}(s) = \sum_{n=1}^{\infty} \Lambda(n)a_n n^{-s},$$

where Λ is the von Mangoldt function and $a_n = 0$ unless n is prime or a prime power. Then by [10, Lemma 5.2], we have

$$\sum_{n \le x} \Lambda(n)|a_n|^2 \sim x \quad \text{as } x \to \infty. \tag{1.2}$$

By the estimate of Kim and Sarnak [9], we have $|a_n| \le n^{7/64} + n^{-7/64}$, so the contribution of composite n to (1.2) is $O(x^{\frac{23}{32}})$. Since $a_q = \lambda_f(q)$ for primes q, this implies that

$$\sum_{\substack{q \text{ prime} \\ q \le x}} (\log q)|\lambda_f(q)|^2 \sim x,$$

and (1.1) follows by partial summation and the prime number theorem.

In particular, there are infinitely many $q \nmid N$ such that $|\lambda_f(q)| < 2$. For any such q, it follows that $D_f(s,\chi_0)$ has infinitely many poles on the line $\Re(s) = 0$. In view of the above, $D_f(s, 1/q, \cos)$ inherits these poles when they occur. On the other hand, under the assumption that $L_f(s)$ has at most finitely many non-trivial simple zeros, we will show that $D_f(s, 1/q, \cos)$ is holomorphic apart from possible poles along two horizontal lines. The contradiction between these two implies the main theorem.

1.1. **Overview.** We begin with an overview of the proof. First, by [6, (4.36)], f has the Fourier–Whittaker expansion

$$f(x+iy) = \sum_{n=1}^{\infty} \Big(\rho(n)W_{\frac{k}{2},\nu}(4\pi ny)e(nx) + \rho(-n)W_{-\frac{k}{2},\nu}(4\pi ny)e(-nx)\Big),$$

where $W_{\alpha,\beta}$ is the Whittaker function defined in [6, (4.20)], and $\nu = \sqrt{\frac{1}{4} - \lambda}$, where λ is the eigenvalue of f with respect to the weight k Laplace operator. When $k = 1$, the Selberg eigenvalue conjecture holds, so that $\nu \in i[0,\infty)$. When $k = 0$ the conjecture remains open, but we have the partial result of Kim–Sarnak [9] that $\nu \in (0, \frac{7}{64}] \cup i[0,\infty)$.

Since f is primitive, it is an eigenfunction of the operator Q_{sk} defined in [6, (4.65)], so that

$$\rho(-n) = \epsilon \frac{\Gamma(\frac{1+k}{2} + \nu)}{\Gamma(\frac{1-k}{2} + \nu)} \rho(n) = \epsilon \nu^k \rho(n)$$

for some $\epsilon \in \{\pm 1\}$. Further, we have $\rho(n) = \rho(1)\lambda_f(n)/\sqrt{n}$. Choosing the normalization $\rho(1) = \pi^{-\frac{k}{2}}$ and writing $e(\pm nx) = \cos(2\pi nx) \pm i\sin(2\pi nx)$, we obtain the expansion

$$f(x+iy) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} \big(V_f^+(ny)\cos(2\pi nx) + iV_f^-(ny)\sin(2\pi nx)\big), \tag{1.3}$$

where

$$V_f^{\pm}(y) = \pi^{-\frac{k}{2}}\left(W_{\frac{k}{2},\nu}(4\pi y) \pm \epsilon\nu^k W_{-\frac{k}{2},\nu}(4\pi y)\right) = \begin{cases} 4\sqrt{y}K_\nu(2\pi y) & \text{if } k=0 \text{ and } \epsilon=\pm 1,\\ 0 & \text{if } k=0 \text{ and } \epsilon = \mp 1,\\ 4yK_{\nu\pm\frac{\epsilon}{2}}(2\pi y) & \text{if } k=1.\end{cases} \tag{1.4}$$

Let $\bar{f}(z) := \overline{f(-\bar{z})}$ denote the dual of f. Since f is primitive, it is also an eigenfunction of the operator $\overline{W}_k$ defined in [6, (6.10)], so we have

$$f(z) = \eta\left(i\frac{|z|}{z}\right)^k \bar{f}\left(-\frac{1}{Nz}\right) \tag{1.5}$$

for some $\eta \in \mathbb{C}$ with $|\eta| = 1$.

Next we define a formal Fourier series $F(z)$ associated to $D_f(s)$ by replacing $\lambda_f(n)$ in the above by $c_f(n)$:

$$F(x+iy) = \sum_{n=1}^{\infty}\frac{c_f(n)}{\sqrt{n}}\big(V_f^+(ny)\cos(2\pi nx) + iV_f^-(ny)\sin(2\pi nx)\big).$$

We expect $F(z)$ to satisfy a relation similar to the modularity relation (1.5). To make this precise, we first recall the functional equation for $L_f(s)$. Define

$$\gamma_f^{\pm}(s) = \Gamma_{\mathbb{R}}\left(s + \frac{1\mp(-1)^k\epsilon}{2} + \nu\right)\Gamma_{\mathbb{R}}\left(s + \frac{1\mp\epsilon}{2} - \nu\right). \tag{1.6}$$

Then the complete L-function $\Lambda_f(s) := \gamma_f^+(s)L_f(s)$ satisfies

$$\Lambda_f(s) = \eta\epsilon^{1-k}N^{\frac{1}{2}-s}\Lambda_{\bar{f}}(1-s), \tag{1.7}$$

with η as above.

We define a completed version of $D_f(s)$ by multiplying by the same Γ-factor: $\Delta_f(s) := \gamma_f^+(s)D_f(s)$. Then, differentiating the functional equation (1.7), we obtain

$$\Delta_f(s) + \big(\psi_f'(s) - \psi_{\bar{f}}'(1-s)\big)\Lambda_f(s) = \eta\epsilon^{1-k}N^{\frac{1}{2}-s}\Delta_{\bar{f}}(1-s), \tag{1.8}$$

where $\psi_f(s) := \frac{d}{ds}\log\gamma_f^+(s)$. In Section 2, we take a suitable inverse Mellin transform of (1.8). Under the assumption that $\Lambda_f(s)$ has at most finitely many simple zeros, this yields a pseudo-modularity relation for F of the form

$$F(z) + A(z) = \eta\left(i\frac{|z|}{z}\right)^k \overline{F}\left(-\frac{1}{Nz}\right) + B(z), \tag{1.9}$$

for certain auxiliary functions A and B, where $\overline{F}(z) := \overline{F(-\bar{z})}$. Roughly speaking, A is the contribution from the correction term $\big(\psi_f'(s) - \psi_{\bar{f}}'(1-s)\big)\Lambda_f(s)$ in (1.8), and B comes from the non-trivial poles of $\Delta_f(s)$.

The main technical ingredient needed to carry this out is the following pair of Mellin transforms involving the K-Bessel function and trigonometric functions [7, 6.699(3) and 6.699(4)]:

(1.10)
$$\int_0^\infty x^{\lambda+1}K_\mu(ax)\sin(bx)\frac{dx}{x} = 2^\lambda b\Gamma\left(\frac{2+\lambda+\mu}{2}\right)\Gamma\left(\frac{2+\lambda-\mu}{2}\right){}_2\mathrm{F}_1\left(\frac{2+\lambda+\mu}{2},\frac{2+\lambda-\mu}{2};\frac{3}{2};-\frac{b^2}{a^2}\right)$$

and

(1.11)
$$\int_0^\infty x^{\lambda+1}K_\mu(ax)\cos(bx)\frac{dx}{x} = \frac{2^{\lambda-1}}{a^{\lambda+1}}\Gamma\left(\frac{1+\lambda+\mu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right){}_2\mathrm{F}_1\left(\frac{1+\lambda+\mu}{2},\frac{1+\lambda-\mu}{2};\frac{1}{2};-\frac{b^2}{a^2}\right),$$

where

$${}_2\mathrm{F}_1(a,b;c;z) = \sum_{j=1}^{\infty}\frac{a(a+1)\cdots(a+j-1)\cdot b(b+1)\cdots(b+j-1)}{c(c+1)\cdots(c+j-1)}\frac{z^j}{j!} \tag{1.12}$$

is the Gauss hypergeometric function. The origin of these hypergeometric factors is explained in the introduction to [3], and the need to analyze them is the main difference between this paper and the holomorphic case from [1] (for which corresponding factors are elementary functions).

Specializing (1.9) to $z=\alpha+iy$ for $\alpha\in\mathbb{Q}^\times$, we have

$$F(\alpha+iy)+A(\alpha+iy)=\eta\left(i\frac{|\alpha+iy|}{\alpha+iy}\right)^k\overline{F}\left(-\frac{1}{N(\alpha+iy)}\right)+B(\alpha+iy). \tag{1.13}$$

We will take the Mellin transform of (1.13). Without difficulty the reader can guess that the transform of $F(\alpha+iy)$ will be a combination of $D_f(s,\alpha,\cos)$ and $D_f(s,\alpha,\sin)$. The calculation of the other terms is non-trivial, but ultimately we obtain the following proposition, which will play the role of Proposition 2.1 in [1]:

Proposition 1.2. *Suppose that $\Lambda_f(s)$ has at most finitely many simple zeros. Then, for every $M\in\mathbb{Z}_{\geq 0}$ and $a\in\{0,1\}$,*

$$P_f(s;a,0)\Delta_f(s,\alpha,\cos^{(a+k)})$$
$$-\eta(-\operatorname{sgn}\alpha)^k(N\alpha^2)^{s-\frac12}\sum_{m=0}^{M-1}\frac{(2\pi N\alpha)^m}{m!}P_f(s;a,m)\Delta_{\bar f}\left(s+m,-\frac{1}{N\alpha},\cos^{(a+m)}\right)$$

is holomorphic for $\Re(s)>\frac32-M$ except for possible poles for $s\pm\nu\in\mathbb{Z}$, where

$$P_f(s;a,m)=\frac{\gamma_f^{(-)^a}(1-s)}{\gamma_f^{(-)^a}(1-s-2\lfloor m/2\rfloor)}\begin{cases}\frac{s+2\lfloor m/2\rfloor-(-1)^a\epsilon\nu}{2\pi} & \text{if } k=1 \text{ and } 2\nmid m,\\ 0 & \text{if } k=0 \text{ and } (-1)^a=-\epsilon,\\ 1 & \text{otherwise}\end{cases}$$

and

$$\Delta_f(s,\alpha,\cos^{(a)})=\gamma_f^{(-)^a}(s)D_f(s,\alpha,\cos^{(a)}).$$

1.2. **Proof of Theorem 1.1.** Assuming Proposition 1.2 for the moment, we can complete the proof of Theorem 1.1 for the case of π corresponding to a Maass cusp form, f. First, as noted above, we may choose a prime $q\nmid N$ for which $D_f(s,1/q,\cos)$ has infinitely many poles on the line $\Re(s)=0$. Then, by Dirichlet's theorem on primes in an arithmetic progression, for any $M\in\mathbb{Z}_{>0}$ there are distinct primes $q_0,q_1,\ldots,q_{M-1}$ such that $q_j\equiv q \pmod N$ and $D_{\bar f}(s,-q_j/N,\cos^{(a)})=D_{\bar f}(s,-q/N,\cos^{(a)})$ for all j, a.

Let m_0 be an integer with $0\leq m_0\leq M-1$. By the Vandermonde determinant, there exist rational numbers $c_0,c_1,\ldots,c_{M-1}$ such that

$$\sum_{j=0}^{M-1}c_jq_j^{-m}=\begin{cases}1 & \text{if } m=m_0,\\ 0 & \text{if } m\neq m_0\end{cases}\quad\text{for all } m\in\{0,1,\ldots,M-1\}.$$

We fix $\delta\in\{0,1\}$ and apply Proposition 1.2 with $a\equiv\delta+m_0 \pmod 2$ and $\alpha=1/q_j$ for $j=0,1,\ldots,M-1$. Multiplying by $(-1)^kc_j(q_j^2/N)^{s-\frac12}$, summing over j and replacing s by $s-m_0$, we find that

$$\sum_{j=0}^{M-1}(-1)^kc_j\left(\frac{q_j^2}{N}\right)^{s-m_0-\frac12}P_f(s-m_0;\delta+m_0,0)\Delta_f\left(s-m_0,\frac{1}{q_j},\cos^{(\delta+m_0+k)}\right)$$
$$-\eta\frac{(-2\pi N)^{m_0}}{m_0!}P_f(s-m_0;\delta+m_0,m_0)\Delta_{\bar f}\left(s,-\frac{q}{N},\cos^{(\delta)}\right)$$

is holomorphic on $\{s\in\Omega:\Re(s)>\frac32+m_0-M\}$, where we set

$$\Omega=\{s\in\mathbb{C}:s\pm\nu\notin\mathbb{Z}\}.$$

Since $D_f(s-m_0,1/q_j,\cos^{(\delta+m_0+k)})$ is holomorphic on $\{s\in\Omega:\Re(s)<m_0-\frac12\}$, choosing $m_0=2+\delta+\frac{1-\epsilon}{2}$ and M arbitrarily large, we conclude that $D_{\bar f}(s,-q/N,\cos^{(\delta)})$ is holomorphic on Ω.

Next we apply Proposition 1.2 again with $a=k$, $\alpha=1/q$ and $M=2$. When $k=1$ or $k=0$ and $\epsilon=1$, we see that $D_f(s,1/q,\cos)$ is holomorphic on $\{s\in\Omega:\Re(s)=0\}$. This is a contradiction, and Theorem 1.1 follows in these cases.

The remaining case is that of odd Maass forms of weight 0. The above argument with $\delta=1$ shows that $D_f(s,-q/N,\sin)$ is entire apart from possible poles for $s\pm\nu\in\mathbb{Z}$. Applying Proposition 1.2 with $a=1$, $\alpha=-q/N$ and $M=3$, we find that

$$-\Delta_f\left(s,-\frac{q}{N},\sin\right)+\eta\left(\frac{q^2}{N}\right)^{s-\frac12}\left[\Delta_{\bar f}\left(s,\frac1q,\sin\right)-2\pi q\Delta_{\bar f}\left(s+1,\frac1q,\cos\right)\right.$$
$$\left.-\frac{(2\pi q)^2}{2!}P_f(s;1,2)\Delta_{\bar f}\left(s+2,\frac1q,\sin\right)\right]$$

is holomorphic on $\{s \in \Omega : \Re(s) > -\frac{5}{2}\}$. Since $D_{\bar f}(s, 1/q, \sin)$ is holomorphic on the lines $\Re(s) = -1$ and $\Re(s) = 1$, we see that $D_{\bar f}(s, 1/q, \cos)$ is holomorphic on $\{s \in \Omega : \Re(s) = 0\}$. This is again a contradiction, and concludes the proof.

2. Proof of Proposition 1.2

Using the expansion (1.3), we take the Mellin transform of (1.5) along the line $z = (\omega + i)y$. First, the left-hand side becomes, for $\Re(s) \gg 1$,

$$\begin{aligned}\int_0^\infty f(\omega y + iy) y^{s-\frac{1}{2}} \frac{dy}{y} &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}} \int_0^\infty \big(V_f^+(ny)\cos(2\pi n\omega y) + iV_f^-(ny)\sin(2\pi n\omega y)\big) y^{s-\frac{1}{2}} \frac{dy}{y} \\ &= G_f(s,\omega) L_f(s),\end{aligned} \tag{2.1}$$

where, by (1.4), (1.10) and (1.11),
(2.2)

$$\begin{aligned} G_f(s,\omega) &= \int_0^\infty \big(V_f^+(y)\cos(2\pi\omega y) + iV_f^-(y)\sin(2\pi\omega y)\big) y^{s-\frac{1}{2}}\frac{dy}{y} \\ &= \begin{cases} (2\pi i\omega)^{\frac{1-\epsilon}{2}} \gamma_f^+(s)\, {}_2\mathrm{F}_1\Big(\frac{s+\frac{1-\epsilon}{2}+\nu}{2}, \frac{s+\frac{1-\epsilon}{2}-\nu}{2}; 1-\frac{\epsilon}{2}; -\omega^2\Big) & \text{if } k=0, \\ \gamma_f^+(s)\, {}_2\mathrm{F}_1\Big(\frac{s+\frac{1+\epsilon}{2}+\nu}{2}, \frac{s+\frac{1-\epsilon}{2}-\nu}{2}; \frac{1}{2}; -\omega^2\Big) + 2\pi i\omega\gamma_f^-(s+1)\, {}_2\mathrm{F}_1\Big(\frac{s+\frac{3-\epsilon}{2}+\nu}{2}, \frac{s+\frac{3+\epsilon}{2}-\nu}{2}; \frac{3}{2}; -\omega^2\Big) & \text{if } k=1. \end{cases}\end{aligned}$$

Note that we have $G_{\bar f}(s,\omega) = \overline{G_f(\bar s, -\omega)}$.

On the other hand, the Mellin transform of the right-hand side of (1.5) is, for $-\Re(s) \gg 1$,

$$\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k \int_0^\infty \bar f\Big(-\frac{\omega}{N(\omega^2+1)y} + \frac{i}{N(\omega^2+1)y}\Big) y^{s-\frac{1}{2}}\frac{dy}{y}.$$

Making the substitution $y \mapsto (N(\omega^2+1)y)^{-1}$, this becomes
(2.3)

$$\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k \big(N(1+\omega^2)\big)^{\frac{1}{2}-s} \int_0^\infty \bar f(-\omega y + iy) y^{\frac{1}{2}-s}\frac{dy}{y} = \eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k \big(N(1+\omega^2)\big)^{\frac{1}{2}-s} G_{\bar f}(1-s,-\omega) L_{\bar f}(1-s).$$

By (1.5), (2.1) and (2.3) must continue to entire functions and equal each other. In particular, taking $\omega \to 0$, we recover the functional equation (1.7). Equating (2.1) with (2.3) and dividing by (1.7), we discover the functional equation for the hypergeometric factor $H_f(s,\omega) := G_f(s,\omega)/\gamma_f^+(s)$:

$$H_f(s,\omega) = \epsilon^{1-k}\left(i\frac{|\omega+i|}{\omega+i}\right)^k (1+\omega^2)^{\frac{1}{2}-s} H_{\bar f}(1-s,-\omega). \tag{2.4}$$

Next, for $z = x+iy \in \mathbb{H}$, define

$$A(z) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \big(\psi'(s+\nu) + \psi'(s-\nu)\big) H_f(s, x/y)\Lambda_f(s) y^{\frac{1}{2}-s}\,ds$$

and

$$B(z) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{1}{2}} X_f(s)\Lambda_f(s) H_f(s,x/y) y^{\frac{1}{2}-s}\,ds - \sum_\rho \Lambda_f'(\rho) H_f(\rho, x/y) y^{\frac{1}{2}-\rho}, \tag{2.5}$$

where the sum runs over all simple zeros of $\Lambda_f(s)$, and

$$X_f(s) = \frac{\pi^2}{4}\left[\csc^2\!\Big(\frac{\pi}{2}\Big[s + \frac{1+(-1)^k\epsilon}{2} + \nu\Big]\Big) + \csc^2\!\Big(\frac{\pi}{2}\Big[s+\frac{1+\epsilon}{2} - \nu\Big]\Big)\right].$$

Lemma 2.1.

$$F(z) + A(z) = \eta\left(i\frac{|z|}{z}\right)^k \overline{F}\Big(-\frac{1}{Nz}\Big) + B(z) \quad \textit{for all } z \in \mathbb{H}.$$

Proof. Fix $z = x+iy \in \mathbb{H}$, and put $\omega = x/y$. Applying Mellin inversion as in (2.1), we have

$$F(z) = \frac{1}{2\pi i}\int_{\Re(s)=2} D_f(s) G_f(s,\omega) y^{\frac{1}{2}-s}\,ds$$

and

$$\eta\left(i\frac{|z|}{z}\right)^k\overline{F}\left(-\frac{1}{Nz}\right)=\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k\cdot\frac{1}{2\pi i}\int_{\Re(s)=2}G_{\bar f}(s,-\omega)D_{\bar f}(s)\big(N(1+\omega^2)y\big)^{s-\frac12}\,ds$$
$$=\eta\left(i\frac{|\omega+i|}{\omega+i}\right)^k\cdot\frac{1}{2\pi i}\int_{\Re(s)=-1}H_{\bar f}(1-s,-\omega)\Delta_{\bar f}(1-s)\big(N(1+\omega^2)y\big)^{\frac12-s}\,ds.$$

Applying 2.4 and (1.8), and using the fact that $\psi'_{\bar f}(1-s)$ is holomorphic for $\Re(s)\le\frac12$, the last line becomes

$$\frac{1}{2\pi i}\int_{\Re(s)=-1}\eta\epsilon^{1-k}H_f(s,\omega)\Delta_{\bar f}(1-s)(Ny)^{\frac12-s}\,ds$$
$$=\frac{1}{2\pi i}\int_{\Re(s)=-1}H_f(s,\omega)\Big[\Delta_f(s)+\big(\psi'_f(s)-\psi'_{\bar f}(1-s)\big)\Lambda_f(s)\Big]y^{\frac12-s}\,ds$$
$$=\frac{1}{2\pi i}\int_{\Re(s)=-1}H_f(s,\omega)\Big[\Delta_f(s)+\psi'_f(s)\Lambda_f(s)\Big]y^{\frac12-s}\,ds-\frac{1}{2\pi i}\int_{\Re(s)=\frac12}H_f(s,\omega)\psi'_{\bar f}(1-s)\Lambda_f(s)y^{\frac12-s}\,ds.$$

Shifting the contour of the first integral to the right and using that $\psi'_f(s)$ is holomorphic for $\Re(s)\ge\frac12$, we get

$$\frac{1}{2\pi i}\int_{\Re(s)=2}H_f(s,\omega)\Delta_f(s)y^{\frac12-s}\,ds-\frac{1}{2\pi i}\int_{\mathcal C}H_f(s,\omega)\big(\Delta_f(s)+\psi'_f(s)\Lambda_f(s)\big)y^{\frac12-s}\,ds$$
$$+\frac{1}{2\pi i}\int_{\Re(s)=\frac12}\big(\psi'_f(s)-\psi'_{\bar f}(1-s)\big)H_f(s,\omega)\Lambda_f(s)y^{\frac12-s}\,ds,$$

where $\mathcal C$ is the contour running from $2-i\infty$ to $2+i\infty$ and from $-1+i\infty$ to $-1-i\infty$. Note that

$$\Delta_f(s)+\psi'_f(s)\Lambda_f(s)=\Lambda_f(s)\frac{d^2}{ds^2}\log\Lambda_f(s),$$

which has a pole at every simple zero ρ of $\Lambda_f(s)$, with residue $-\Lambda'_f(\rho)$. Hence,

$$-\frac{1}{2\pi i}\int_{\mathcal C}H_f(s,\omega)\big(\Delta_f(s)+\psi'_f(s)\Lambda_f(s)\big)y^{\frac12-s}\,ds=\sum_\rho\Lambda'_f(\rho)H_f(\rho,\omega)y^{\frac12-\rho}.$$

Next, writing $\psi_{\mathbb R}(s)=\frac{\Gamma'_{\mathbb R}}{\Gamma_{\mathbb R}}(s)$, we have

$$\psi_f(s)=\psi_{\mathbb R}\left(s+\frac{1-(-1)^k\epsilon}{2}+\nu\right)+\psi_{\mathbb R}\left(s+\frac{1-\epsilon}{2}-\nu\right).$$

Applying the reflection formula and Legendre duplication formula in the form

$$\psi'_{\mathbb R}(s)=\frac{\pi^2}{4}\csc^2\Big(\frac{\pi s}{2}\Big)-\psi'_{\mathbb R}(2-s)\quad\text{and}\quad\psi'_{\mathbb R}(s)+\psi'_{\mathbb R}(s+1)=\psi'(s),$$

we derive

$$\psi'_f(s)-\psi'_{\bar f}(1-s)=\psi'(s+\nu)+\psi'(s-\nu)-X_f(s).$$

Thus,

$$\frac{1}{2\pi i}\int_{\Re(s)=\frac12}\big(\psi'_f(s)-\psi'_{\bar f}(1-s)\big)H_f(s,\omega)\Lambda_f(s)y^{\frac12-s}\,ds=A(z)-\frac{1}{2\pi i}\int_{\Re(s)=\frac12}X_f(s)H_f(s,\omega)\Lambda_f(s)y^{\frac12-s}\,ds.$$

Rearranging terms completes the proof. ☐

Lemma 2.2. *For any $\alpha\in\mathbb Q^\times$,*

$$\frac{1}{\Gamma(s+\nu)\Gamma(s-\nu)}\int_0^\infty A(\alpha+iy)y^{s-\frac12}\frac{dy}{y}$$

continues to an entire function of s.

Proof. Define $\Phi(s) = \psi'(s+\nu) + \psi'(s-\nu)$. Then we have $\Phi(s) = \int_1^\infty \phi(x) x^{\frac12 - s}\,dx$, where $\phi(x) = \frac{\cosh(\nu \log x)\log x}{\sinh(\frac12 \log x)}$. Applying (2.2) and the change of variables $y \mapsto xt$, we have

$$\begin{aligned}\Phi(s)G_f(s,\omega) &= \int_1^\infty \int_0^\infty \phi(x)\big(V_f^+(y)\cos(2\pi\omega y) + iV_f^-(y)\sin(2\pi\omega y)\big)\left(\frac{y}{x}\right)^{s-\frac12}\frac{dy}{y}\,dx\\ &= \int_0^\infty \left(\int_1^\infty \phi(x)\big(V_f^+(tx)\cos(2\pi\omega tx) + iV_f^-(tx)\sin(2\pi\omega tx)\big)\,dx\right) t^{s-\frac12}\frac{dt}{t}.\end{aligned}$$

Hence, writing $\omega = \alpha/y$, we have

$$\begin{aligned}A(\alpha+iy) &= \frac{1}{2\pi i}\int_{\Re(s)=2} \Lambda_f(s)\Phi(s)H_f(s,\omega)y^{\frac12 - s}\,ds = \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}}\frac{1}{2\pi i}\int_{\Re(s)=2}\Phi(s)G_f(s,\omega)(ny)^{\frac12-s}\,ds\\ &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}}\int_1^\infty \phi(x)\big(V_f^+(nxy)\cos(2\pi\alpha nx) + iV_f^-(nxy)\sin(2\pi\alpha nx)\big)\,dx,\end{aligned}$$

so that

$$\begin{aligned}\int_0^\infty A(\alpha+iy)y^{s-\frac12}\frac{dy}{y} &= \sum_{n=1}^\infty \frac{\lambda_f(n)}{\sqrt{n}}\int_1^\infty \phi(x)\int_0^\infty \big(V_f^+(nxy)\cos(2\pi\alpha nx) + iV_f^-(nxy)\sin(2\pi\alpha nx)\big)y^{s-\frac12}\frac{dy}{y}\,dx\\ &= \sum_{n=1}^\infty \lambda_f(n)n^{-s}\int_1^\infty \phi(x)x^{\frac12-s}\Big(\widetilde{V}_f^+(s)\cos(2\pi\alpha nx) + i\widetilde{V}_f^-(s)\sin(2\pi\alpha nx)\Big)\,dx,\end{aligned}$$

where

$$\widetilde{V}_f^\pm(s) = \int_0^\infty V_f^\pm(y)y^{s-\frac12}\frac{dy}{y} = \begin{cases}\gamma_f^\pm(s) & \text{if } k=1 \text{ or } \epsilon = \pm 1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.6}$$

A case-by-case inspection of (1.6) shows that $\widetilde{V}_f^\pm(s)/(\Gamma(s+\nu)\Gamma(s-\nu))$ is entire for both choices of sign.

Define $\phi_j = \phi_j(x,s)$ for $j \ge 0$ by

$$\phi_0 = \phi, \quad \text{and} \quad \phi_{j+1} = x\frac{\partial \phi_j}{\partial x} - (s+j-\tfrac12)\phi_j.$$

Then, applying integration by parts m times, we see that

$$\int_1^\infty \phi(x)\cos(2\pi\alpha nx)x^{\frac12-s}\,dx = \sum_{j=0}^{m-1}\frac{\cos^{(j+1)}(2\pi\alpha n)}{(2\pi\alpha n)^{j+1}}\phi_j(1,s) + \int_1^\infty \frac{\cos^{(m)}(2\pi\alpha nx)}{(2\pi\alpha n)^m}\phi_k(x,s)x^{\frac12-m-s}\,dx$$

and

$$\int_1^\infty \phi(x)\sin(2\pi\alpha nx)x^{\frac12-s}\,dx = \sum_{j=0}^{m-1}\frac{\sin^{(j+1)}(2\pi\alpha n)}{(2\pi\alpha n)^{j+1}}\phi_j(1,s) + \int_1^\infty \frac{\sin^{(m)}(2\pi\alpha nx)}{(2\pi\alpha n)^m}\phi_k(x,s)x^{\frac12-m-s}\,dx.$$

Thus,

$$\begin{aligned}&\int_0^\infty A(\alpha+iy)y^{s-\frac12}\frac{dy}{y}\\ &= \widetilde{V}_f^+(s)\left[\sum_{j=0}^{m-1}\frac{\phi_j(1,s)L(f,s+j+1,\alpha,\cos^{(j+1)})}{(2\pi\alpha)^{j+1}} + \frac{1}{(2\pi\alpha)^m}\sum_{n=1}^\infty \frac{a_f(n)}{n^{s+m}}\int_1^\infty \cos^{(m)}(2\pi\alpha nx)\phi_m(x,s)x^{\frac12-m-s}\,dx\right]\\ &+ i\widetilde{V}_f^-(s)\left[\sum_{j=0}^{m-1}\frac{\phi_j(1,s)L(f,s+j+1,\alpha,\sin^{(j+1)})}{(2\pi\alpha)^{j+1}} + \frac{1}{(2\pi\alpha)^m}\sum_{n=1}^\infty \frac{a_f(n)}{n^{s+m}}\int_1^\infty \sin^{(m)}(2\pi\alpha nx)\phi_m(x,s)x^{\frac12-m-s}\,dx\right].\end{aligned}$$

It follows from [2, Prop. 3.1] that $L_f(s,\alpha,\cos)$ and $L_f(s,\alpha,\sin)$ continue to entire functions. We see by induction that $\phi_m(x,s) \ll_m \big((1+|s|)(1+|\nu|)\big)^m x^{-1}$ uniformly for $x \ge 1$, and thus the integral terms above are holomorphic for $\Re(s) > \frac12 - m$. Choosing m arbitrarily large, the lemma follows. $\square$

Lemma 2.3. *For any $\sigma \ge 0$ and any $l \in \mathbb{Z}_{\ge 0}$, we have*

$$\frac{y^l}{l!}(V_{\bar f}^\pm)^{(l)}(y) \ll_\sigma 2^l y^{-\sigma} \quad \textit{for } y > 0.$$

Proof. In view of (2.6), since $|\Re(\nu)| < \frac{1}{2}$, for any $\sigma \geq 0$ we have the integral representation

$$V_{\bar{f}}^{\pm}(y) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma+\frac{1}{2}} \widetilde{V}_{\bar{f}}^{\pm}(s) y^{\frac{1}{2}-s}\, ds.$$

Differentiating l times, we obtain

$$\frac{y^l}{l!}(V_{\bar{f}}^{\pm})^{(l)}(y) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma+\frac{1}{2}} \binom{\frac{1}{2}-s}{l} \widetilde{V}_{\bar{f}}^{\pm}(s) y^{\frac{1}{2}-s}\, ds.$$

Using the estimate

$$\left|\binom{\frac{1}{2}-s}{l}\right| = \left|\binom{s-\frac{1}{2}+l}{l}\right| \leq 2^{|s-\frac{1}{2}|+l},$$

we have

$$\frac{y^l}{l!}(V_{\bar{f}}^{\pm})^{(l)}(y) \leq 2^l y^{-\sigma} \cdot \frac{1}{2\pi} \int_{\Re(s)=\sigma+\frac{1}{2}} 2^{|s-\frac{1}{2}|} \big|\widetilde{V}_{\bar{f}}^{\pm}(s)\, ds\big| \ll_\sigma 2^l y^{-\sigma},$$

where the last inequality is justified by Stirling's formula. $\square$

Lemma 2.4. *Let $\alpha \in \mathbb{Q}^\times$ and $z = \alpha + iy$ for some $y \in (0, |\alpha|/2]$. Then, for any integer $T \geq 0$, we have*

(2.7)

$$\left(i\frac{|z|}{z}\right)^k \overline{F}\left(-\frac{1}{Nz}\right) = O_{\alpha,T}(y^{T-1}) + (i\operatorname{sgn}(\alpha))^k \sum_{t=0}^{T-1} \frac{(2\pi i N\alpha)^t}{t!}$$
$$\cdot \sum_{a\in\{0,1\}} \frac{i^{-a}}{2\pi i} \int_{\Re(s)=2} P_f(s; a+t, t) \Delta_{\bar{f}}\left(s+t, -\frac{1}{N\alpha}, \cos^{(a)}\right) \left(\frac{y}{N\alpha^2}\right)^{\frac{1}{2}-s} ds.$$

Proof. Let $z = \alpha + iy$, $\beta = -1/N\alpha$ and $u = y/\alpha$. Then

$$-\frac{1}{Nz} = \frac{\beta}{1+u^2} + i\frac{|\beta u|}{1+u^2},$$

so that

$$\begin{aligned}
\left(i\frac{|z|}{z}\right)^k \overline{F}\left(-\frac{1}{Nz}\right) &= \left(i\operatorname{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k \overline{F}\left(\frac{\beta}{1+u^2} + i\frac{|\beta u|}{1+u^2}\right) \\
= \left(i\operatorname{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k &\sum_{n=1}^{\infty} \frac{c_{\bar{f}}(n)}{\sqrt{n}} \left(V_{\bar{f}}^{+}\left(\frac{|\beta n u|}{1+u^2}\right) \cos\left(\frac{2\pi\beta n}{1+u^2}\right) + iV_{\bar{f}}^{-}\left(\frac{|\beta n u|}{1+u^2}\right) \sin\left(\frac{2\pi\beta n}{1+u^2}\right)\right).
\end{aligned}$$

By Lemma 2.3, for any $\sigma \geq 0$ and any $l_0 \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned}
V_{\bar{f}}^{\pm}\left(\frac{|\beta n u|}{1+u^2}\right) &= \sum_{l=0}^{\infty} \frac{1}{l!}(V_{\bar{f}}^{\pm})^{(l)}(|\beta n u|) \left(\frac{\beta n u^3}{1+u^2}\right)^l \\
&= \sum_{l=0}^{l_0-1} \frac{1}{l!}(V_{\bar{f}}^{\pm})^{(l)}(|\beta n u|) \left(\frac{\beta n u^3}{1+u^2}\right)^l + O_\sigma\left(|\beta n u|^{-\sigma} \sum_{l=l_0}^{\infty} \left(\frac{2u^2}{1+u^2}\right)^l\right) \\
&= \sum_{l=0}^{l_0-1} \frac{1}{l!}(V_{\bar{f}}^{\pm})^{(l)}(|\beta n u|) \left(\frac{\beta n u^3}{1+u^2}\right)^l + O_{\alpha,\sigma,l_0}\left(|nu|^{-\sigma} u^{2l_0}\right).
\end{aligned}$$

Similarly, for any $a \in \{0,1\}$, we have

$$\begin{aligned}
\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right) &= \sum_{j=0}^{\infty} \frac{1}{j!} \cos^{(j+a)}(2\pi\beta n) \left(-\frac{2\pi\beta n u^2}{1+u^2}\right)^j \\
&= \sum_{j=0}^{j_0-1} \frac{1}{j!} \cos^{(j+a)}(2\pi\beta n) \left(-\frac{2\pi\beta n u^2}{1+u^2}\right)^j + O\left(\frac{1}{j_0!}\left|\frac{2\pi\beta n u^2}{1+u^2}\right|^{j_0}\right) \\
&= \sum_{j=0}^{j_0-1} \frac{1}{j!} \cos^{(j+a)}(2\pi\beta n) \left(-\frac{2\pi\beta n u^2}{1+u^2}\right)^j + O_{\alpha,j_0}\left((nu^2)^{j_0}\right),
\end{aligned}$$

by the Lagrange form of the error in Taylor's theorem. Taking $j_0 = 2(l_0 - l)$ and applying Lemma 2.3 with σ replaced by $\sigma + 2(l_0 - l)$, we obtain

$$
\begin{aligned}
&V_{\bar{f}}^{(-)^a}\left(\frac{|\beta n u|}{1+u^2}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right)\\
&\qquad = \sum_{j+2l<2l_0} \frac{(-2\pi)^j}{j!l!}(V_{\bar{f}}^{(-)^a})^{(l)}(|\beta n u|)\cos^{(j+a)}(2\pi\beta n)u^l\left(\frac{\beta n u^2}{1+u^2}\right)^{j+l} + O_{\alpha,\sigma,l_0}\big(|nu|^{-\sigma}u^{2l_0}\big).
\end{aligned}
$$

Next, defining

$$
b_{j,k,l,m} = \begin{cases} \binom{j+l-1+\lfloor \frac{m}{2}\rfloor + \frac{k}{2}}{\lfloor \frac{m}{2}\rfloor} & \text{if } k=1 \text{ or } k=0 \text{ and } 2\mid m,\\ 0 & \text{otherwise,}\end{cases}
$$

we have

$$
\begin{aligned}
\left(\frac{|1+iu|}{1+iu}\right)^k (1+u^2)^{-j-l} &= (1-iu)^k(1+u^2)^{-j-l-\frac{k}{2}} = \sum_{m=0}^{\infty} b_{j,k,l,m}(-iu)^m\\
&= \sum_{m=0}^{m_0-1} b_{j,k,l,m}(-iu)^m + O\left(\sum_{m=m_0}^{\infty} 2^{j+l+\frac{m}{2}}|u|^m\right)\\
&= \sum_{m=0}^{m_0-1} b_{j,k,l,m}(-iu)^m + O_{j,l,m_0}(|u|^{m_0}).
\end{aligned}
$$

Taking $m_0 = 2l_0 - j - 2l$ and applying Lemma 2.3 with σ replaced by $\sigma + j$, we obtain

$$
\begin{aligned}
&\left(i\,\mathrm{sgn}(\alpha)\frac{|1+iu|}{1+iu}\right)^k V_{\bar{f}}^{(-)^a}\left(\frac{|\beta n|}{1+u^2}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+u^2}\right)\\
&= (i\,\mathrm{sgn}(\alpha))^k \sum_{j+2l+m<2l_0} \frac{(-2\pi)^j(-i)^m}{j!l!} b_{j,k,l,m}(\beta n u)^{j+l}(V_{\bar{f}}^{(-)^a})^{(l)}(|\beta n u|)\cos^{(j+a)}(2\pi\beta n)u^{j+2l+m}\\
&\quad + O_{\alpha,\sigma,l_0}\big(|nu|^{-\sigma}u^{2l_0}\big).
\end{aligned}
$$

Recalling the definition of u, multplying by $c_{\bar{f}}(n)/\sqrt{n}$ and summing over n and both choices of a, the error term converges if $\sigma \geq 1$, to give

$$
\begin{aligned}
&\sum_{a\in\{0,1\}} i^{-a}\left(i\frac{|\alpha+iy|}{\alpha+iy}\right)^k \sum_{n=1}^{\infty}\frac{c_{\bar{f}}(n)}{\sqrt{n}} V_{\bar{f}}^{(-)^a}\left(\frac{ny}{N(\alpha^2+y^2)}\right)\cos^{(a)}\left(\frac{2\pi\beta n}{1+(y/\alpha)^2}\right)\\
&= \sum_{j+2l+m<2l_0}(i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}\sum_{n=1}^{\infty}\frac{c_{\bar{f}}(n)}{\sqrt{n}}\frac{(2\pi i)^j}{j!l!} b_{j,k,l,m}\left(\frac{ny}{N\alpha^2}\right)^{j+l}\\
&\qquad\qquad \cdot (V_{\bar{f}}^{(-)^a})^{(l)}\left(\frac{ny}{N\alpha^2}\right)\cos^{(j+a)}(2\pi\beta n)\left(\frac{y}{i\alpha}\right)^{j+2l+m} + O_{\alpha,\sigma,l_0}\big(y^{2l_0-\sigma}\big)\\
&= \sum_{j+2l+m<2l_0}(i\,\mathrm{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}\sum_{n=1}^{\infty}\frac{c_{\bar{f}}(n)}{\sqrt{n}}\frac{(-2\pi)^j}{j!l!} b_{j,k,l,m}\left(\frac{ny}{N\alpha^2}\right)^{j+l}\\
&\qquad\qquad \cdot (V_{\bar{f}}^{(-)^{a+j}})^{(l)}\left(\frac{ny}{N\alpha^2}\right)\cos^{(a)}(2\pi\beta n)\left(\frac{y}{i\alpha}\right)^{j+2l+m} + O_{\alpha,\sigma,l_0}\big(y^{2l_0-\sigma}\big).
\end{aligned}
$$

Taking the Mellin transform of a single term of the sum over j, l, m and making the change of variables $y \mapsto N\alpha^2 y/n$, we get

$$
\begin{aligned}
&(i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a} \int_0^\infty \sum_{n=1}^\infty \frac{c_{\bar f}(n)}{\sqrt{n}} \frac{(-2\pi)^j}{j!l!} b_{j,k,l,m} \left(\frac{ny}{N\alpha^2}\right)^{j+l} \\
&\qquad\qquad \cdot \left(V_{\bar f}^{(-)^{a+j}}\right)^{(l)}\left(\frac{ny}{N\alpha^2}\right) \cos^{(a)}(2\pi\beta n) \left(\frac{y}{i\alpha}\right)^{j+2l+m} y^{s-\frac12}\frac{dy}{y} \\
&= (i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}(N\alpha^2)^{s-\frac12}(-iN\alpha)^{j+2l+m}\frac{(-2\pi)^j}{j!} b_{j,k,l,m} \\
&\qquad\qquad \cdot \sum_{n=1}^\infty \frac{c_{\bar f}(n)\cos^{(a)}(2\pi\beta n)}{n^{s+j+2l+m}} \int_0^\infty \frac{y^l}{l!}\left(V_{\bar f}^{(-)^{a+j}}\right)^{(l)}(y) y^{s+2j+2l+m-\frac12}\frac{dy}{y} \\
&= (i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}(N\alpha^2)^{s-\frac12}(-iN\alpha)^{t}\frac{(-2\pi)^j}{j!} b_{j,k,l,m} \\
&\qquad\qquad \cdot D_{\bar f}(s+t,\beta,\cos^{(a)}) \binom{\frac12 - s - t - j}{l} \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j),
\end{aligned}
$$

where $t = j + 2l + m$.

Next we fix $t \in \mathbb{Z}_{\geq 0}$ and sum over all (j, l, m) satisfying $j + 2l + m = t$. When $k = 0$, $b_{j,k,l,m}$ vanishes unless m is even. Hence, defining

$$
I_k(m) = \begin{cases} 1 & \text{if } k = 1 \text{ or } 2 \mid m, \\ 0 & \text{otherwise,} \end{cases}
$$

we get

$$
\begin{aligned}
&(i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}(N\alpha^2)^{s-\frac12}(-iN\alpha)^{t} \sum_{j+2l+m=t} I_k(t-j)\frac{(-2\pi)^j}{j!} \binom{j+l-1+\lfloor\frac{m}{2}\rfloor+\frac{k}{2}}{\lfloor\frac{m}{2}\rfloor} \\
&\qquad\qquad \cdot D_{\bar f}(s+t,\beta,\cos^{(a)}) \binom{\frac12 - s - t - j}{l} \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \\
&= (i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}(N\alpha^2)^{s-\frac12}(-iN\alpha)^{t} \sum_{j=0}^{t} I_k(t-j)\frac{(-2\pi)^j}{j!} D_{\bar f}(s+t,\beta,\cos^{(a)}) \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \\
&\qquad\qquad \cdot \sum_{l=0}^{\lfloor\frac{t-j}{2}\rfloor} \binom{j+\lfloor\frac{t-j}{2}\rfloor+\frac{k}{2}-1}{\lfloor\frac{t-j}{2}\rfloor - l}\binom{\frac12 - s - t - j}{l} \\
&= (i\operatorname{sgn}(\alpha))^k \sum_{a\in\{0,1\}} i^{-a}(N\alpha^2)^{s-\frac12}(-iN\alpha)^{t} \sum_{j=0}^{t} I_k(t-j)\frac{(-2\pi)^j}{j!} D_{\bar f}(s+t,\beta,\cos^{(a)}) \widetilde{V}_{\bar f}^{(-)^{a+j}}(s+t+j) \\
&\qquad\qquad \cdot \binom{\lfloor\frac{t-j}{2}\rfloor+\frac{k-1}{2}-s-t}{\lfloor\frac{t-j}{2}\rfloor},
\end{aligned}
$$

by the Chu–Vandermonde identity.

We now break into cases according to the weight, k. When $k = 0$, the inner sum vanishes identically when $(-1)^{a+t} = -\epsilon$, so we may assume that $(-1)^{a+t} = \epsilon$. Thus, in this case, we have

$$
(N\alpha^2)^{s-\frac12}(iN\alpha)^t i^{-a} D_{\bar f}(s+t,\beta,\cos^{(a)}) \sum_{\substack{j\leq t \\ j\equiv t \pmod 2}} \frac{(2\pi)^j}{j!}\gamma_{\bar f}^{(-)^{a+t}}(s+t+j)\binom{\frac{t-j}{2}-\frac12-s-t}{\frac{t-j}{2}}.
$$

Put $t = 2n + b$, with $b \in \{0, 1\}$. Then, writing $j = 2r + b$, the above becomes

$$
\begin{aligned}
&(N\alpha^2)^{s-\frac{1}{2}}(iN\alpha)^t i^{-a} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)}) \\
&\quad \cdot \sum_{r=0}^{n} \frac{(2\pi)^{2r+b}}{(2r+b)!} \frac{\Gamma_{\mathbb{R}}(s+t+2r+b+\nu)\Gamma_{\mathbb{R}}(s+t+2r+b-\nu)}{\Gamma_{\mathbb{R}}(s+t+b+\nu)\Gamma_{\mathbb{R}}(s+t+b-\nu)} \binom{n-r-\frac{1}{2}-s-t}{n-r} \\
&= (N\alpha^2)^{s-\frac{1}{2}}(iN\alpha)^t i^{-a} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)})(-1)^n \\
&\quad \cdot \sum_{r=0}^{n} \left(\frac{2\pi}{2r+1}\right)^b \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t+b+\nu)/2}{r} \binom{-(s+t+b-\nu)/2}{r} \binom{s+t-\frac{1}{2}}{n-r}.
\end{aligned}
$$

Applying [2, Lemma A.1(ii)–(iii)], we get

$$
\begin{aligned}
&(N\alpha^2)^{s-\frac{1}{2}}(iN\alpha)^t i^{-a} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)}) \\
&\quad \cdot \left(\frac{2\pi}{2n+1}\right)^b \frac{4^n n!^2}{(2n)!} \binom{(s+t-1-b+\nu)/2}{n} \binom{(s+t-1-b-\nu)/2}{n} \\
&= (N\alpha^2)^{s-\frac{1}{2}} \frac{(2\pi i N\alpha)^t}{t!} i^{-a} \frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2n)} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)}).
\end{aligned}
$$

Turning to $k = 1$, we have

$$
\begin{aligned}
&i \operatorname{sgn}(\alpha)(N\alpha^2)^{s-\frac{1}{2}}(-iN\alpha)^t \sum_{a\in\{0,1\}} i^{-a} \sum_{j=0}^{t} \frac{(-2\pi)^j}{j!} D_{\bar{f}}(s+t, \beta, \cos^{(a)}) \\
&\qquad\qquad \cdot \gamma_{\bar{f}}^{(-)^{a+j}}(s+t+j) \binom{\lfloor\frac{t-j}{2}\rfloor - s - t}{\lfloor\frac{t-j}{2}\rfloor} \\
&= i \operatorname{sgn}(\alpha)(N\alpha^2)^{s-\frac{1}{2}}(-iN\alpha)^t \sum_{a\in\{0,1\}} i^{-a} D_{\bar{f}}(s+t, \beta, \cos^{(a)}) \\
&\qquad\qquad \cdot \sum_{j=0}^{t} \frac{(-2\pi)^j}{j!} \gamma_{\bar{f}}^{(-1)^{a+j}}(s+t+j) \binom{\lfloor\frac{t-j}{2}\rfloor - s - t}{\lfloor\frac{t-j}{2}\rfloor}.
\end{aligned}
$$

Writing $j = 2r - c$ with $c \in \{0, 1\}$, this is

$$
\begin{aligned}
&i \operatorname{sgn}(\alpha)(N\alpha^2)^{s-\frac{1}{2}}(-iN\alpha)^t \sum_{a\in\{0,1\}} i^{-a} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)}) \sum_{c\in\{0,1\}} \sum_{2r-c\le t} \frac{(-2\pi)^{2r-c}}{(2r-c)!} \binom{n-r+\lfloor\frac{b+c}{2}\rfloor - s - t}{n-r+\lfloor\frac{b+c}{2}\rfloor} \\
&\quad \cdot \frac{\Gamma_{\mathbb{R}}\left(s+t+2r-c+\frac{1-(-1)^{a+c}\epsilon}{2}+\nu\right)\Gamma_{\mathbb{R}}\left(s+t+2r-c+\frac{1+(-1)^{a+c}\epsilon}{2}-\nu\right)}{\Gamma_{\mathbb{R}}\left(s+t+\frac{1-(-1)^{a}\epsilon}{2}+\nu\right)\Gamma_{\mathbb{R}}\left(s+t+\frac{1+(-1)^{a}\epsilon}{2}-\nu\right)} \\
&= i \operatorname{sgn}(\alpha)(N\alpha^2)^{s-\frac{1}{2}}(-iN\alpha)^t \sum_{a\in\{0,1\}} i^{-a} \Delta_{\bar{f}}(s+t, \beta, \cos^{(a)}) \sum_{c\in\{0,1\}} (-1)^{n+bc} \\
&\quad \cdot \sum_{2r-c\le t} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r-c\frac{1-(-1)^a\epsilon}{2}} \binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r-c\frac{1+(-1)^a\epsilon}{2}} \binom{s+t-1}{n+bc-r}.
\end{aligned}
$$

For $b = 0$, applying [2, Lemma A.1(ii)], the sum over c becomes

$$
\begin{aligned}
&(-1)^n \sum_{r=0}^{n} \frac{(-4)^r r!^2}{(2r)!} \binom{-(s+t-1+\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{r} \binom{-(s+t-1+\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{r} \binom{s+t-1}{n-r} \\
&= \frac{4^n n!^2}{(2n)!} \binom{(s+2n-2+\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{n} \binom{(s+2n-2+\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{n} \\
&= \frac{(-2\pi)^{2n}}{(2n)!} \frac{\Gamma_{\mathbb{R}}(1-s+\frac{1+(-1)^a\epsilon}{2}+\nu)}{\Gamma_{\mathbb{R}}(1-s-2n+\frac{1+(-1)^a\epsilon}{2}+\nu)} \frac{\Gamma_{\mathbb{R}}(1-s+\frac{1-(-1)^a\epsilon}{2}-\nu)}{\Gamma_{\mathbb{R}}(1-s-2n+\frac{1-(-1)^a\epsilon}{2}-\nu)} \\
&= \frac{(-2\pi)^t}{t!} \frac{\gamma_f^{(-)^a}(1-s)}{\gamma_f^{(-)^a}(1-s-2n)} = \frac{(-2\pi)^t}{t!} \frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2\lfloor t/2\rfloor)}.
\end{aligned}
$$

For $b=1$ and $c=0$, the inner sum is

$$(-1)^n\sum_{r=0}^{n}\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r}\binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r}\binom{s+t-1}{n-r}.$$

Writing $\binom{s+t-1}{n-r}=\binom{s+t}{n-r+1}-\binom{s+t-1}{n-r+1}$ and applying [2, Lemma A.1(ii)], we get

$$\begin{aligned}
&(-1)^n\Bigg[\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{(s+t-\frac{1+(-1)^a\epsilon}{2}+\nu)/2}{n+1}\binom{(s+t-\frac{1-(-1)^a\epsilon}{2}-\nu)/2}{n+1}\\
&\qquad-\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n+1}\binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n+1}\Bigg]\\
&+(-1)^{n+1}\Bigg[\sum_{r=0}^{n+1}\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{r}\binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{r}\binom{s+t-1}{n-r+1}\\
&\qquad-\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{-(s+t+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n+1}\binom{-(s+t+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n+1}\Bigg]\\
&=(-1)^n\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1}\binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1}\\
&\quad+(-1)^{n+1}\sum_{r=0}^{n+1}\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t-(-1)^a\epsilon\nu+1)/2}{r}\binom{-(s+t+(-1)^a\epsilon\nu)/2}{r}\binom{s+t-1}{n-r+1}.
\end{aligned}$$

For $b=1$ and $c=1$ the inner sum is

$$(-1)^{n+1}\sum_{r=1}^{n+1}\frac{(-4)^r r!^2}{(2r)!}\binom{-(s+t-(-1)^a\epsilon\nu+1)/2}{r-1}\binom{-(s+t+(-1)^a\epsilon\nu)/2}{r}\binom{s+t-1}{n+1-r},$$

and adding this to the contribution from $c=0$, for $b=1$ we obtain

$$\begin{aligned}
&(-1)^n\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1}\binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1}\\
&\quad+(-1)^{n+1}\left[\binom{s+t-1}{n+1}+\sum_{r=1}^{n+1}\frac{(-4)^r r!^2}{(2r)!}\binom{1-(s+t-(-1)^a\epsilon\nu+1)/2}{r}\binom{-(s+t+(-1)^a\epsilon\nu)/2}{r}\binom{s+t-1}{n-r+1}\right]\\
&=(-1)^n\frac{(-4)^{n+1}(n+1)!^2}{(2n+2)!}\binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1}\binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1}\\
&\quad+(-1)^{n+1}\sum_{r=0}^{n+1}\frac{(-4)^r r!^2}{(2r)!}\binom{1-(s+t-(-1)^a\epsilon\nu+1)/2}{r}\binom{-(s+t+(-1)^a\epsilon\nu)/2}{r}\binom{s+t-1}{n-r+1}.
\end{aligned}$$

Applying [2, Lemma A.1(ii)], this is

$$\begin{aligned}
&-\frac{4^{n+1}(n+1)!^2}{(2n+2)!}\binom{(s+t-1+(-1)^a\epsilon\nu)/2}{n+1}\left[\binom{(s+t-(-1)^a\epsilon\nu)/2}{n+1}-\binom{(s+t-(-1)^a\epsilon\nu)/2-1}{n+1}\right]\\
&=-\frac{4^{n+1}(n+1)!^2}{(2n+2)!}\frac{(s+(-1)^a\epsilon\nu+2n)/2}{n+1}\binom{(s+2n-2+\frac{1+(-1)^a\epsilon}{2}-\nu)/2}{n}\binom{(s+2n-2+\frac{1-(-1)^a\epsilon}{2}+\nu)/2}{n}\\
&=\frac{s+2\lfloor t/2\rfloor-(-1)^{a+t}\epsilon\nu}{2\pi}\frac{(-2\pi)^t}{t!}\frac{\gamma_f^{(-)^{a+t}}(1-s)}{\gamma_f^{(-)^{a+t}}(1-s-2\lfloor t/2\rfloor)}.
\end{aligned}$$

In all cases, the result matches the formula for $P_f(s;a+t,t)$. Taking $l_0=\lceil T/2\rceil$, $\sigma=1$ and applying Mellin inversion, we get (2.7), with $T+1$ in place of T when T is odd. In that case, we estimate the final term of the sum by shifting the contour to $\Re(s)=\frac{3}{2}-T$, which yields $O(y^{T-1})$. ☐

Lemma 2.5. *Assume that $\Lambda_f(s)$ has at most finitely many simple zeros, and let $\alpha\in\mathbb{Q}^\times$ and $z=\alpha+iy$ for some $y\in(0,|\alpha|/4]$. Then there are numbers $a_j(\alpha),b_j(\alpha)\in\mathbb{C}$ such that, for any integer $M\geq 0$, we have*

$$B(\alpha+iy)=O_{\alpha,f,M}(y^M)+\sum_{j=0}^{M-1}y^{j+\frac12}\begin{cases}a_j(\alpha)+b_j(\alpha)\log y & \text{if } \nu=k=0,\\ a_j(\alpha)y^\nu+b_j(\alpha)y^{-\nu} & \text{otherwise.}\end{cases}\tag{2.8}$$

Proof. Let $s \in \mathbb{C}$ with $\Re(s) \in (0,1)$, and set $\omega = \alpha/y$. We will show that there are numbers $a_j(\alpha,s), b_j(\alpha,s) \in \mathbb{C}$ satisfying

$$H_f(s,\omega)y^{\frac{1}{2}-s} = \sum_{j=0}^{\infty} y^{j+\frac{1}{2}} \begin{cases} a_j(\alpha,s) + b_j(\alpha,s)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha,s)y^{\nu} + b_j(\alpha,s)y^{-\nu} & \text{otherwise} \end{cases} \tag{2.9}$$

and

$$a_j(\alpha,s), b_j(\alpha,s) \ll_{f,\alpha,\varepsilon} (2e^{\pi/2})^{(1+\varepsilon)|s|}|2/\alpha|^{j+\frac{1}{2}}\sqrt{j+1}, \quad \text{for all } \varepsilon > 0. \tag{2.10}$$

Let us assume this for now. Then, since $y \le |\alpha|/4$, we have

$$\sum_{j=M}^{\infty}\left(\frac{2y}{|\alpha|}\right)^{j+\frac{1}{2}}\sqrt{j+1} \ll_{\alpha,M} y^{M+\frac{1}{2}},$$

so that (by the trivial estimate $|\Re(\nu)| < \frac{1}{2}$),

$$H_f(s,\omega)y^{\frac{1}{2}-s} = O_{f,\alpha,M,\varepsilon}((2e^{\pi/2})^{(1+\varepsilon)|s|}y^M) + \sum_{j=0}^{M-1} y^{j+\frac{1}{2}} \begin{cases} a_j(\alpha,s) + b_j(\alpha,s)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha,s)y^{\nu} + b_j(\alpha,s)y^{-\nu} & \text{otherwise.} \end{cases} \tag{2.11}$$

We substitute this expansion into (2.5). By hypothesis, $\Lambda_f(s)$ has at most finitely many simple zeros, so the sum over ρ in (2.5) is a finite linear combination of the series (2.11) with $s = \rho$, which yields an expansion of the shape (2.8). As for the integral term in (2.5), by the convexity bound and Stirling's formula, we have

$$X_f(s)\Lambda_f(s) \ll_{f,\varepsilon} e^{-(3\pi/2-\varepsilon)|s|} \quad \text{for } \Re(s) = \tfrac{1}{2}, \varepsilon > 0.$$

Since $2 < e^{\pi}$, the integral converges absolutely and again yields something of the shape (2.8).

It remains to show (2.9) and (2.10). First suppose that $k = 0$. Then, by (2.2), we have

$$H_f(s,\omega)y^{\frac{1}{2}-s} = |\alpha/\omega|^{\frac{1}{2}-s}(2\pi i\omega)^{\frac{1-\epsilon}{2}}\,{}_2\mathrm{F}_1\left(\frac{s+\frac{1-\epsilon}{2}+\nu}{2}, \frac{s+\frac{1-\epsilon}{2}-\nu}{2}; 1-\frac{\epsilon}{2}; -\omega^2\right).$$

Applying the hypergeometric transformation [7, 9.132(2)] and the defining series (1.12), this is

$$(\pi i\operatorname{sgn}(\alpha))^{\frac{1-\epsilon}{2}}|\alpha|^{\frac{1}{2}-s}\pi^{\frac{1}{2}}\sum_{\pm}\frac{|y/\alpha|^{\frac{1}{2}\pm\nu}\Gamma(\mp\nu)}{\Gamma\left(1-\frac{s+\frac{1+\epsilon}{2}\pm\nu}{2}\right)\Gamma\left(\frac{s+\frac{1-\epsilon}{2}\mp\nu}{2}\right)}\,{}_2\mathrm{F}_1\left(\frac{s+\frac{1-\epsilon}{2}\pm\nu}{2}, \frac{s+\frac{1+\epsilon}{2}\pm\nu}{2}; 1\pm\nu; -\left(\frac{y}{\alpha}\right)^2\right) \tag{2.12}$$

$$= (\pi i\operatorname{sgn}(\alpha))^{\frac{1-\epsilon}{2}}|\alpha|^{\frac{1}{2}-s}\pi^{\frac{1}{2}}\sum_{j=0}^{\infty}\sum_{\pm}\frac{\Gamma(\mp\nu)}{\Gamma\left(1-\frac{s+\frac{1+\epsilon}{2}\pm\nu}{2}\right)\Gamma\left(\frac{s+\frac{1-\epsilon}{2}\mp\nu}{2}\right)}\frac{\binom{-\frac{s+\frac{1-\epsilon}{2}\pm\nu}{2}}{j}\binom{-\frac{s+\frac{1+\epsilon}{2}\pm\nu}{2}}{j}}{\binom{-1\mp\nu}{j}}\left|\frac{y}{\alpha}\right|^{2j+\frac{1}{2}\pm\nu}.$$

To pass from this to (2.9), we replace $2j$ by j and set $a_j = b_j = 0$ when j is odd.

When $\nu \ne 0$ we use the estimates

$$\left|\binom{-\frac{s+a\pm\nu}{2}}{j}\right| = \left|\binom{\frac{s+a\pm\nu}{2}+j-1}{j}\right| \le 2^{|s+a\pm\nu|/2+j} \ll_f 2^{|s|/2+j} \quad \text{for } a \in \{0,1\},$$

$$\left|\binom{-1\mp\nu}{j}\right| = \prod_{l=1}^{j}\left|1\pm\frac{\nu}{l}\right| \ge \prod_{l=1}^{j}\left|1-\frac{1}{2l}\right| = \left|\binom{-\frac{1}{2}}{j}\right| \gg \frac{1}{\sqrt{2j+1}}$$

and

$$(\pi i\operatorname{sgn}(\alpha))^{\frac{1-\epsilon}{2}}|\alpha|^{\frac{1}{2}-s}\pi^{\frac{1}{2}}\frac{\Gamma(\mp\nu)}{\Gamma\left(1-\frac{s+\frac{1+\epsilon}{2}\pm\nu}{2}\right)\Gamma\left(\frac{s+\frac{1-\epsilon}{2}\mp\nu}{2}\right)} \ll_{f,\varepsilon} e^{(\pi/2+\varepsilon)|s|} \quad \text{for all } \varepsilon > 0$$

to obtain (2.10).

When $\nu = 0$, (2.12) has a singularity arising from the $\Gamma(\pm\nu)$ factors, but we can still understand the formula by analytic continuation. To remove the singularity, we replace $y^{\pm\nu}$ by $(y^{\pm\nu}-1)+1$. Since

$$\lim_{\nu\to 0}\Gamma(\pm\nu)(y^{\pm\nu}-1) = \log y,$$

in the terms with $y^{\pm\nu}-1$ we can simply take the limit and estimate the remaining factors as before; this gives the b_j terms in (2.9) and (2.10). The terms with 1 can be written in the form $y^{2j+\frac{1}{2}}(h_j(\nu)+h_j(-\nu))$, where h_j is meromorphic with a simple pole at $\nu = 0$, and independent of y. Then $h_j(\nu) + h_j(-\nu)$ is even, so it has a removable singularity at $\nu = 0$. By the Cauchy integral formula, we have

$$\lim_{\nu\to 0}(h_j(\nu) + h_j(-\nu)) = \frac{1}{2\pi i}\int_{|\nu|=\frac{1}{2}} \frac{h_j(\nu) + h_j(-\nu)}{\nu}\, d\nu.$$

Since the above estimates hold uniformly for $\nu \in \mathbb{C}$ with $|\nu| = \frac{1}{2}$, they also hold for $\lim_{\nu\to 0}(h_j(\nu) + h_j(-\nu))$. This concludes the proof of (2.9) and (2.10) when $k = 0$.

Turning to $k = 1$, by (2.2) we have

$$\begin{aligned} H_f(s,\omega)y^{\frac{1}{2}-s} = \sum_{\delta\in\{0,1\}} \left|\frac{\alpha}{\omega}\right|^{\frac{1}{2}-s} (i\omega(s-\epsilon\nu))^{\delta} \\ \cdot {}_2\mathrm{F}_1\left(\frac{s+(-1)^{\delta}\frac{1+\epsilon}{2}+\nu}{2}+\delta, \frac{s+(-1)^{\delta}\frac{1-\epsilon}{2}-\nu}{2}+\delta; \frac{1}{2}+\delta; -\omega^2\right), \end{aligned}$$

and applying [7, 9.132(2)], this becomes

$$\begin{aligned} \pi^{\frac{1}{2}}|\alpha|^{\frac{1}{2}-s} \sum_{\delta\in\{0,1\}} \left(\frac{i\,\mathrm{sgn}(\alpha)(s-\epsilon\nu)}{2}\right)^{\delta} \sum_{\pm} \left|\frac{y}{\alpha}\right|^{\frac{1}{2}+\frac{1\pm(-1)^{\delta}\epsilon}{2}\pm\nu} \frac{\Gamma\big(\mp(\nu+(-1)^{\delta}\frac{\epsilon}{2})\big)}{\Gamma\big(\frac{s+(-1)^{\delta}\frac{1\mp\epsilon}{2}\mp\nu}{2}+\delta\big)\Gamma\big(\frac{1}{2}-\frac{s+(-1)^{\delta}\frac{1\pm\epsilon}{2}\pm\nu}{2}\big)} \\ \cdot {}_2\mathrm{F}_1\left(\frac{s+(-1)^{\delta}\frac{1\pm\epsilon}{2}\pm\nu}{2}+\delta, \frac{s+(-1)^{\delta}\frac{1\pm\epsilon}{2}\pm\nu}{2}+\frac{1}{2}; 1\pm\left(\nu+(-1)^{\delta}\frac{\epsilon}{2}\right); -\left(\frac{y}{\alpha}\right)^2\right). \end{aligned}$$

In this case no singularity arises from the Γ-factor in the numerator, so expanding the final ${}_2\mathrm{F}_1$ as a series and applying a similar analysis to the above, we arrive at (2.9) and (2.10). □

With the lemmas in place, we can now complete the proof of Proposition 1.2. Let

$$\chi_{(0,\frac{|\alpha|}{4}]}(y) = \begin{cases} 1 & \text{if } y \le \frac{|\alpha|}{4}, \\ 0 & \text{if } y > \frac{|\alpha|}{4}, \end{cases}$$

and define

$$\begin{aligned} g(y) = F(\alpha+iy) + A(\alpha+iy) - \chi_{(0,\frac{|\alpha|}{4}]}(y)\sum_{j=0}^{M-1} y^{j+\frac{1}{2}} \begin{cases} a_j(\alpha) + b_j(\alpha)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha)y^{\nu} + b_j(\alpha)y^{-\nu} & \text{otherwise} \end{cases} \\ - \eta(i\,\mathrm{sgn}(\alpha))^k \sum_{t=0}^{M-1} \frac{(2\pi i N\alpha)^t}{t!} \sum_{a\in\{0,1\}} \frac{i^{-a}}{2\pi i}\int_{\Re(s)=2} P_f(s;a+t,t)\Delta_{\bar{f}}\left(s+t, -\frac{1}{N\alpha}, \cos^{(a)}\right)\left(\frac{y}{N\alpha^2}\right)^{\frac{1}{2}-s} ds. \end{aligned}$$

By Lemmas 2.1, 2.4 and 2.5, we have $g(y) = O_{\alpha,M}(y^{M-1})$ for $y \le |\alpha|/4$. On the other hand, shifting the contour of the above to the right, we see that g decays rapidly as $y \to \infty$. Hence, $\int_0^\infty g(y)y^{s-\frac{1}{2}}\frac{dy}{y}$ converges absolutely and defines a holomorphic function for $\Re(s) > \frac{5}{2} - M$.

We have

$$\int_0^\infty F(\alpha+iy)y^{s-\frac{1}{2}}\frac{dy}{y} = \sum_{a\in\{0,1\}} i^{-a}\Delta_f\big(s,\alpha,\cos^{(a)}\big) \begin{cases} 1 & \text{if } k=1 \text{ or } (-1)^a = \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.2, $\int_0^\infty A(\alpha+iy)y^{s-\frac{1}{2}}\frac{dy}{y}$ continues to a holomorphic function on Ω. Similarly,

$$\begin{aligned} \int_0^\infty y^{s-\frac{1}{2}}\frac{dy}{y}\cdot\chi_{(0,\frac{|\alpha|}{4}]}(y)\sum_{j=0}^{M-1} y^{j+\frac{1}{2}} \begin{cases} a_j(\alpha) + b_j(\alpha)\log y & \text{if } \nu = k = 0, \\ a_j(\alpha)y^{\nu} + b_j(\alpha)y^{-\nu} & \text{otherwise} \end{cases} \\ = \sum_{j=0}^{M-1} \begin{cases} \frac{|\alpha/4|^{s+j}}{s+j}\left[a_j(\alpha) + b_j(\alpha)\left(\log|\alpha/4| - \frac{1}{s+j}\right)\right] & \text{if } \nu = k = 0, \\ a_j(\alpha)\frac{|\alpha/4|^{s+j+\nu}}{s+j+\nu} + b_j(\alpha)\frac{|\alpha/4|^{s+j-\nu}}{s+j-\nu} & \text{otherwise} \end{cases} \end{aligned}$$

is holomorphic on Ω. Hence, by Mellin inversion,

$$(2.13)\quad \sum_{a\in\{0,1\}} i^{-a}\Delta_f\big(s,\alpha,\cos^{(a)}\big)\begin{cases}1 & \text{if } k=1 \text{ or } (-1)^a=\epsilon,\\ 0 & \text{otherwise}\end{cases}$$
$$-\,\eta(i\operatorname{sgn}(\alpha))^k(N\alpha^2)^{s-\frac12}\sum_{t=0}^{M-1}\frac{(2\pi iN\alpha)^t}{t!}\sum_{a\in\{0,1\}} i^{-a}P_f(s;a+t,t)\Delta_{\bar f}\left(s+t,-\frac{1}{N\alpha},\cos^{(a)}\right)$$

is holomorphic on $\{s\in\Omega:\Re(s)>\frac52-M\}$.

Denoting (2.13) by $h(\alpha)$, we consider the combination $\frac12(i^{k+a_0}h(\alpha)+i^{-k-a_0}h(-\alpha))$ for some $a_0\in\{0,1\}$. This picks out the term with $a\equiv k+a_0 \pmod 2$ in the first sum over a, and $a\equiv t+a_0 \pmod 2$ in the second. Therefore, since

$$P_f(s;a_0,0)=\begin{cases}1 & \text{if } k=1 \text{ or } (-1)^{a_0}=\epsilon,\\ 0 & \text{otherwise,}\end{cases}$$

we find that

$$(2.14)\quad P_f(s;a_0,0)\Delta_f\big(s,\alpha,\cos^{(k+a_0)}\big)$$
$$-\,\eta(-\operatorname{sgn}(\alpha))^k(N\alpha^2)^{s-\frac12}\sum_{t=0}^{M-1}\frac{(2\pi N\alpha)^t}{t!}P_f(s;a_0,t)\Delta_{\bar f}\left(s+t,-\frac{1}{N\alpha},\cos^{(t+a_0)}\right)$$

is holomorphic on $\{s\in\Omega:\Re(s)>\frac52-M\}$. Finally, replacing M by $M+1$ and discarding the final term of the sum, we see that (2.14) is holomorphic on $\{s\in\Omega:\Re(s)>\frac32-M\}$, as required.

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